MATHEMATICS IN THE SECONDARY SCHOOLS OF GERMANY, I

The school reforms consequent upon Prussia's defeat in the Napoleonic wars mark the beginning of serious mathematical study in the *Gymnasium*. Hitherto, two to three recitations a week had been considered ample time for a subject so little in harmony with humanistic ideals; ten to fourteen hours weekly were not too many for Latin and Greek. But in the programme for 1816 mathematics was made a main subject alongside of the ancient classics and of equal worth with them. The course led up to and included theory of equations, chance, the elements of analytic geometry, and mechanics. In zeal to outdo France the reformers had been too radical for the schoolmen. In 1827 the time was reduced from six hours a week to four. The programmes of 1837 and 1856 were still less liberal, allotting to *Quinta*, *Quarta*, and *Tertia* only three periods. In 1882 only two classes were left with so few as three recitations a week; the total week-hours were 34. The programme of 1882 decreased the week-hours for mathematics in the *Realgymnasiums* from 47 to 44 and in 1892 a still further reduction of two hours a week was made. The *Oberrealschulen*, with French and English in place of the classics but with a nine-year course as in the *Gymnasien*, have at present 47 week-hours of mathematics.

Mathematics as taught in the best German schools is a unit. If I refer to the sub-courses independently for convenience's sake, it must be remembered that they are more than parallel—they are interlaced and interwoven to a degree that makes it difficult to separate them. A further difficulty arises from the differences in the secondary schools themselves. They have not the same courses nor a common aim. From a quantitative point of view the *Realgymnasium* stands midway between the two extremes; qualitatively it may be questioned if it does not rank at the head in mathematics. But for the sake of a norm it may be well to take a middle ground. I have chosen, therefore, to describe the course of the *Realgymnasium* in Cassel, Dr. Wittich, direktor,—one of the best schools in the kingdom and renowned as the *Alma*

Mater of Prince Henry of Prussia. For illustrations of method I shall draw freely from my experiences in all sorts and conditions of schools, and it goes without saying that whatever of criticism I may indulge in should not be construed as reflections on the Cassel institution. Indeed, I am obliged to go elsewhere for my material, as during my stay in Cassel the celebration of the twenty-fifth anniversary of the founding of the school was of greater interest to the pupils (—and to the visitor?) than the daily routine of the class room. Its curriculum follows necessarily the last Prussian programme and so far as this discriminates against *Realgymnasien*—and it has cast a cloud over them all—in so far does this school suffer with the rest. For this reason one often finds the best results in mathematics in other German states. Prussia is not Germany by any means in educational matters; yet from force of circumstances the smaller states follow her leadership—though at a respectful distance. The course of 1882, which was willingly adopted in the southern states, gave advantages which non-Prussians are loth to yield, and while the Prussian "reforms" have been followed to a certain extent it has been done under protest. The attitude of the southern leader is happily put in the following words addressed to me in criticism of recent changes: "I cannot bring myself blindly to admire a thing merely for the sake of its coming from Berlin". The Saxon ministry has especially favored the *Realgymnasium* and so have most of the duchies. The Weimar *Realgymnasium* has a most enviable reputation and to its director, Dr. Wernekke, I am indebted for many favors. Here the good points of the German system are to be found at their best and the course is not too much "reformed".

A boy on entering *Sexta* at nine years of age is expected to bring with him from his three-year preparatory course the ability to add, subtract, multiply, and divide simple whole numbers. For the lower grades the Cassel course is as follows:*

SEXTA: *Rechnen*, 4 periods. Extended knowledge of numbers from 1–100, especially division of numbers by smaller numbers and factoring. System of tens. Numeration. Repetition of the four fundamental principles with abstract whole numbers. Weights,

* The Weimar *Realgymnasium* has *five* periods a week in *Sexta* and *Quarta*. By teaching simple rule-of-three in *Sexta* time enough is gained for one hour a week of geometrical object lessons in *Quinta*.

measures and money. Reduction of complex numbers and in connection therewith the simplest tasks in decimal fractions. Text-book, *Uebungsbuch von Böhme*, VIII.

QUINTA: *Rechnen*, 4 periods. Preparations for study of fractions. Common and decimal fractions. Rule-of-three. Text-book, *Böhme's*, IX.

QUARTA: I. *Rechnen*, 2 periods. Review of fractions. Rule-of-three with whole numbers and fractions. Profit and loss. Interest, discount, and partnership. Text-book, *Böhme's*, XII.

II. *Plane Geometry*, 2 periods. Introductory course in object lessons. Angles, parallel lines, triangles, quadrilaterals. Simple constructions. Text-book, *Koppe's Planimetrie.*

Arithmetik, the theory of numbers, includes both reckoning with definite numbers (*Rechnen*) and with numbers in the abstract. *Algebra* is the theory of equations. The work of the lower grades, therefore, is with *Rechnen*, practical arithmetic. The aim is to secure "accuracy and facility in operations with figures" and to lay the foundation for future study. The first desideratum is favored by extraneous circumstances. Classes usually number thirty to forty pupils. Recitation rooms are comparatively small. One black-board, and that a small one behind the teacher's desk, must suffice for the needs of the class. An exceptional arrangement is to have two such boards balanced on pulleys, or a second board mounted on an easel nearer the pupils. There are neither slates nor paper for rough work. All reckoning must be done on the board, in the exercise books which are inspected by the teacher, or—in the head. The last, as the path of least resistance, is followed by the average boy notwithstanding his natural prejudices against thinking for himself. Whether these circumstances be cause or effect, I cannot say, but I suspect they are partly both. The German teacher will tell you, however, that "years ago" it was the custom to assign long lessons to be worked out at home, that to secure a reasonable percentage of correct answers rules and copies were invented, but—he will add—the process was purely mechanical. To-day the ideal is that every step in advance shall be taken in the class rooms, that there shall be but one step at a time and that all shall take that step at the same time. This forbids independent home-study; it limits the master's work to teaching.

A recitation opens with questions rapidly put on the review leading up to the work of the day. Answers must be short, concise, and complete sentences. New principles are developed inductively if possible. A boy goes to the board—why more than one board?—and writes a problem as read to him by another boy or by the teacher. Then more questions to the class. The pupil at the board merely registers the progress; he may be equipped with the rest and if he has suggestions to make he may volunteer in the usual way—by raising the hand—and await the master's recognition. Sometimes if the problem is important, the work is erased and another boy performs the same operation, recounting each step aloud while the class copy it into their exercise books. So goes the hour. The proportion of time given to written work as compared with the questions asked and answered is not far from 1 : 5, so great is the stress put upon oral demonstration. The home-work is of the same nature—generally the identical problems if not already written out; but the task must not take more than half an hour of the pupil's free time. A special exercise to be done at home may be required not oftener than once a month. If new problems are set, all inherent difficulties must be previously closed up and sifted in class. The pupil is not to experiment, nor work in the dark.

From the beginning of the course particular stress is put upon facility in mental calculation. Practice is daily afforded in the ordinary work of the class room, but special drill is given with each lesson in the lower grades. At first simple whole numbers are employed but in *Quinta* the work has so far progressed that numbers of two and three digits are freely used. Such work to be of value must be done quickly; the answer should be ready immediately on conclusion of the statement. From twelve such problems given in one recitation I select at random three:—(1) $4,1+0,9-4,9+0,9=?$; (2) $1,2+3,4\times 10=?$; (3) $0,9+2,4+3,1\div 8=?$ Such tasks are rendered the more difficult by the German way of reading decimals—thus in (1): "Four-comma-one plus naught-comma-nine minus four-comma-nine plus naught-comma-nine," etc. This reading of figures and points in succession, though not expressly sanctioned, seems to be connived at in all parts of Germany from the common schools to the universities; illogical as it may be it seems to the foreigner a sensible reaction

against the laws of the grammarians. The next step brings in the black-board; the problem is written out, thus: (25,5+27,45 +31,55):5=?, but the solution is given orally. How far and in what lines these methods shall be developed lies wholly with the teacher. In geometry, too, there is ample field. A good *Tertianer*, I am told, should be able to demonstrate the Pythagorean proposition in his head, following any designation of lines and angles that may be given him. In arithmetic the multiplication table may be taught as high as the 20's, after which it is comparatively easy to make all necessary combinations. But in explaining his methods to me an excellent teacher made this remark: "It is well that a pupil should be familiar with short methods and be able to reckon rapidly in his head, but it is better to know that 18 times 27 is the same as 18 times 20 plus 18 times 7 than to perform the operation mechanically." Whatever the methods in mental arithmetic may be, the results in most German schools are admirable.

Some of the best schools are making a trial of the so-called "Austrian methods" of reckoning. Illustrations follow (The figures here given I have copied from actual class work; no others were used.):—

(*1*) *Subtraction*—

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 954
-761
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 193
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Boy says, "One and *three* make four (writes 3); six and *nine* make fifteen (writes 9); one (to carry), seven and *one* make nine (writes 1).

(*2*) *Multiplication*—

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225,67 . 875=?
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180536
 157969
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197461,25
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Multiply first by 8, then by 7 and then by 5 adding to this product the partial products already found; write only complete sum in last case. The final operation is as follows: 5×7=35 (write 5); 5×6+3+9=42 (write 2); 5×5+4+6+6=41 (write 1); 5×2+4+9+3=26 (write 6), etc.

(*3*) *Division*—

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427654:145=2949,3.
1376
  715
 1354
   490
  ----
    55 rem.
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First figure of quotient is 2. Then 2×5+7=17 (write 7); 2×4+1 (to carry) +3=12 (write 3); 2×1+1+1=4 (writes 1). Bring down next figure (6) of dividend and continue as before.

(*4*) *Square Root*—

$\sqrt{42 \mid 76 \mid 54} = 653,9.$

6 76:125	
5154:1303	
124500:13069	

The methods, as will be seen, are the same as in division.

After a close inspection of pupils' exercises I cannot say that the chances of error are greater than with the usual way. It reduces the written work to a minimum and decidedly increases the rapidity of computation. It is well worth a trial. Schools which have adopted it show no inclination to go back to the old methods.

The rule-of-three plays an important part in *Quinta* and *Quarta.* The easier problems are stated and solved as follows:

(*a*) *Statement*—

If 25 *kg* of *x* cost 53,45 m. what will 155 *kg* cost?

(*b*) *Solution*—

25 *kg* cost 53,45 m., 155 *kg* cost?

150 " " 320,70 "
5 " " 10,69 "

155 *kg* cost 331,39 m.

Rather more is made of compound proportion than might be expected from the practical nature of the course; the method most frequently used is as follows:

(*a*) *Statement*—

A canal 245 *m* long, 3,3 *m* deep, 7 *m* wide is built by 140 men working 546 days at $7\frac{1}{2}$ hours a day; what is the length of another canal 5 *m* deep, 8,2 *m* wide on which 182 men are employed 324 days working $8\frac{1}{3}$ hrs. a day?

(*b*) *Solution*—

140 men in 546 d. of $7\frac{1}{2}$ h. make $3,3^m$ d. 7^m w. 245^m l.
182 " " 324 " $8\frac{1}{3}$ " " 5 " 8,2 " x.

$$x = \frac{245 \times 182 \times 324 \times 50 \times 33 \times 70}{140 \times 546 \times 45 \times 50 \times 82}$$

Say,—If 245 *m* be done by 140 men, one man will do the 140th part, 182 men will do 182 times as much; viz. in 546 days—hence in 1 day the 546th part, in 324 days 324 times as much; viz. in $7\frac{1}{2}$ hrs. (45-6)—hence in 1-6 hr. the 45th part, in $8\frac{1}{3}$ hrs. (50-6) 50 times as much; viz. 33-10 *m* deep—hence if 1-10 *m* deep 33 times as much, if 50-10 *m* deep the 50th part, etc.

The problems of commercial arithmetic are solved in the same form. Here is a task in interest:

(*a*) *Statement*—

What is the interest on 450 m. for 2 yrs. 3 mos. and 10 ds. at 5 per cent.?

(*b*) *Solution*—

100 m.	give	5	m.	int.	in	1 yr.
450 "	"	22,50	"	"	"	"
450 "	"	45,00	"	"	"	2 yrs.
" "	"	5,625	"	"	"	3 mos.
" "	"	,625	"	"	"	10 ds.
450 m.	give	51,25	m.	int.	in	2 yrs. 3 mos. 10 ds.

The greatest difficulties of the lower grades are in common fractions. But from the start every effort is made to keep within the pupils' sphere. When a boy knows what the division of a unit means the term "fraction" has for him a tangible reality, a definite value. Beyond this limit the German teacher hesitates to go. The theory may best be taught with numbers not too large and as for pure practice there is enough of that in other connections. The main thing is to know the value of a fraction both in concrete terms and in its decimal form and to realize that in its treatment only familiar principles are employed. The prevailing use of the decimal system of weights, measures, and money makes the transition comparatively easy. By far the greater part of the work is done orally, i. e. without book, paper, or blackboard.

The introductory course in geometry is given by most non-Prussian schools in *Quinta*, one period a week. The object is to familiarize the pupils with the essentials of geometrical form—"enough to get them looking at things from a geometrical point of view." The object lessons begin with solids which are handled, described, and measured. Thus arise correct ideas of surfaces, lines, and points and their relations. It is but a step to the drawing of figures, and this in turn forms a basis for the systematic study of plane geometry. Formal theorems are unnecessary. A long series of constructions follows the demonstration of such statements as these: "The base angles of an equilateral triangle are equal"; "The angles of a triangle make two right angles"; "Diagonals of rectangles are equal"; "A tangent stands at right

angles to its radius," etc. Such are the concrete methods of the common schools (*Volksschulen*) and there is a party of school-men, including the Herbartians * who would carry them still further in the secondary schools. Their success is not marked, but to the movement may be accredited certain tendencies which are becoming apparent even in the most conservative circles. The school that would *educate* its pupils, it is said, has no use for what is purely theoretical or abstractly mathematical. The universities are for specialists; the higher schools stand for general culture. "Were Shakespeare, Schiller, and Goethe skilled in logarithms and equations of the third degree?" Mathematics is concerned with the *form* of things, not with their *contents*. The mathematician may be Jew or Gentile, materialist or idealist; it is not *what* he thinks but *how* he thinks that is of concern.

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* See *Schultze's Deutsche Erziehung*, pp. 278–79.